# Fuzzy Probability –Based Modeling the Reliability of Algorithmic Processes

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## Abstract

This paper proposes the method, which allows to assess the fuzzy time and fuzzy probability of correct execution the discrete algorithmic process. The source modeling data are represented as fuzzy numbers, which depend on many influencing factors. Fuzzy logic inference, fuzzy extension principle together the crisp reliability models of algorithmic processes are used for modeling.

## 1. Introduction

Many discrete-behavior systems can be analyzed in a unified framework if combined into a class of so-called algorithmic processes (AP). Typical AP include information processing in computer systems, performance of research or design projects, technological production processes etc.. Each of these processes involves a sequence of operations or jobs unfolding in time whose execution leads to the goal achievement. When designing a specific AP, we need to estimate of the following reliability measures:

- $p_{AP}$ – the probability of correct AP execution; this may be interpreted as the reliability of output information, defect-free quality of the output products, reliability of system functioning;
- $t_{AP}$ - the time or other resources required to execute the AP.

Models to estimate $p_{AP}$ and $t_{AP}$ are widely used in reliability theory of man-machine systems (Gubinskii, 1982, Rotshtein, 1987 and 1990). In these studies, the modeling is based on the theory of semi-Markov processes (Korolyuk and Turbin, 1976) whose states correspond to the operators and logical conditions of the given algorithm. Successful application of AP reliability theory (as well as of classical reliability theory (Kozlov and Ushakov, 1975) ) envisages construction of databases with reliability characteristics of the basic elementary operations. However, new operations do not have ex-post statistical estimates of outcomes under real-life conditions. Complex-system designers are therefore often forced to make decisions on the basis of following expert judgements: *“if the human operator is tired, then the number of errors is approximately doubled”* or *“if the equipment is properly maintained and is operated under appropriate conditions, then the reliability is high”*.

The probabilistic reliability theory (Gubinskii, 1982, Rotshtein, 1987 and 1990) is incapable of utilizing input data expressed in the form of natural-language expert judgements. It is therefore relevant to try and develop a so-called “fuzzy reliability theory of algorithmic processes” (Rotshtein, 1994, Rotshtein and Shtovba, 1997 and 1998), which in addition to the probabilistic apparatus also uses the tools of fuzzy set theory (Zadeh, 1965, Zimmerman, 1996) that can manipulate linguistic expert information.

In this article we propose an approach that extends the probabilistic AP's reliability models to the case of fuzzy input data and allows for the dependence of data on influential factors through fuzzy inference. In terms of fuzzy reliability (Cai, 1996), extended AP's reliability models one can account as a branch of probist reliability theory with fuzzy probabilities.

## 2. Language for description the algorithmic processes

For formal description of AP we use the language of Glushkov's algorithmic algebras (Glushkov, Tseitlin, and Yushchenko, 1978). In this language, the algorithm operators are denoted by Latin capital letters (A, B, C, …) and logical conditions are denoted Greek lower-case letters (α, β, γ, …). By the regularization theorem (Glushkov, Tseitlin, and Yushchenko, 1978), every algorithm is representable a superposition of the following operator structures:

- $B = A_1 A_2$ - linear structure consists of the process of consecutive operators $A_1$ and $A_2$ execution in the order of their registration;
- $C = (A_1 \underset{\alpha}{\vee} A_2)$ - α-disjunction representing operator $A_1$ execution when condition α is true (α=1), and execution of operator $A_2$ when condition α is false (α=0);
- $D = \{A\}_{\alpha}$ - α-iteration representing cyclic execution of operator A till condition α has become true.

## 3. Probabilistic models of algorithm reliability

Let us assume that in execution of any operator A and logical condition $\omega$ the following events are possible: $A^1$ $(A^0)$ – correct (incorrect) execution of operator A*;* $\omega^1$ $(\omega^0)$ – condition $\omega$ is *a priori* true (false); $\omega^{11}$ $(\omega^{10})$ – an *a priori* true condition $\omega$ is recognized as true (false) during a check; $\omega^{00}$ $(\omega^{01})$ – an *a priori* false condition $\omega$ is recognized as false (true) during a check. The above-listed events are assumed pairwise mutually exclusive.The probability ( $Prob$ ) of these events is denoted by:

$$p_A^1 = \text{Prob}\, A^1; \quad p_A^0 = \text{Prob}\, A^0; \quad p_\omega = \text{Prob}\, \omega^1; \quad p_{\bar{\omega}} = \text{Prob}\, \omega^0;$$

$$k_\omega^{11} = \text{Prob}\, \omega^{11}; \quad k_\omega^{10} = \text{Prob}\, \omega^{10}; \quad k_\omega^{00} = \text{Prob}\, \omega^{00}; \quad k_\omega^{01} = \text{Prob}\, \omega^{01}.$$

Note that $k_\omega^{10}$ and $k_\omega^{01}$ are the probabilities of type I and type II errors when checking condition $\omega$. The time for execution the operator A and check the logical condition $\omega$ are denoted by $t_A$ and $t_\omega$.

Error-free execution of operator structures is defined by following logical functions:

$$B^1 = A_1^1 \wedge A_2^1; \quad C^1 = \left(\omega^1 \wedge \omega^{11} \wedge A_1^1\right) \vee \left(\omega^0 \wedge \omega^{00} \wedge A_2^1\right); \quad D^1 = a \vee (b \wedge a) \vee (b \wedge b \wedge a) \vee (b \wedge b \wedge b \wedge a) \ldots ,$$

where $a = A^1 \wedge \omega^{11}$; $b = \left(A^1 \wedge \omega^{10}\right) \vee \left(A^0 \wedge \omega^{00}\right)$.

Given the logical functions of error-free execution of operator structures, we obtain the following rules for estimating the algorithm execution reliability:

$$B = A_1 A_2 \quad \Rightarrow \quad p_B^1 = p_{A_1}^1 \cdot p_{A_2}^1 \;, \quad t_B = t_{A_1} + t_{A_2}; \tag{1}$$

$$C = (A_1 \underset{\omega}{\vee} A_2) \Rightarrow \begin{cases} p_C^1 = p_\omega k_\omega^{11} p_{A_1}^1 + p_{\bar{\omega}} k_\omega^{00} p_{A_2}^1 \\ t_c = t_\omega + \left(p_\omega k_\omega^{11} + p_{\bar{\omega}} k_\omega^{01}\right) t_{A_1} + \left(p_\omega k_\omega^{10} + p_{\bar{\omega}} k_\omega^{00}\right) t_{A2} \end{cases}; \tag{2}$$

$$D=\{A\}_{\omega} \Rightarrow p_D^1=\frac{p_A^1\ k_\omega^{11}}{1-\left(p_A^1\ k_\omega^{10}+p_A^0\ k_\omega^{00}\right)},\quad t_D=\frac{t_A+t_\omega}{1-\left(p_A^1\ k_\omega^{10}+p_A^0\ k_\omega^{00}\right)}. \qquad (3)$$

## 4. Representation of uncertain source data by fuzzy sets

Let q be an uncertain parameter that corresponds to the probability of error-free execution or the cost of executing the operator A or logical condition $\omega$. The uncertain parameter q is treated as a linguistic variable (Zimmerman, 1996) whose levels are formalized by fuzzy sets with convex membership functions defined on the universal set $U=[\underline{q},\overline{q}]$, where $\underline{q}$ and $\overline{q}$ are the smallest and greatest allowed values of the parameter q. In this case, the uncertain parameter q is identified with the fuzzy number $\tilde{q}$. We represent the fuzzy number in following 3 forms: l-, l(X)-, and $\alpha$ - forms.

**Definition 1.** The l-form of the uncertain parameter q is the triple:

$$\tilde{q}=<\underline{q},\overline{q},l>,$$

where l is the linguistic assessment of the parameter q in the range $[\underline{q},\overline{q}]$, selected from the term-set $L=\{l_1,l_2,...,l_m\}$ such that $l_j=\int_U \mu_{l_j}(q)/q$, where $\mu_{l_j}(q)$ is the membership function of the value $q\in[\underline{q},\overline{q}]$ in the term $l_j\in L$, $j=\overline{1,m}$.

**Definition** 2. The $\alpha$ **-**form of the uncertain parameter q is the union of the pairs

$$\tilde{q}=\bigcup_{\alpha\in[0,1]}\left(\underline{q}_\alpha,\overline{q}_\alpha\right), \qquad (4)$$

where $\underline{q}_\alpha(\overline{q}_\alpha)$ is the smallest (greatest) allowed value of q at the $\alpha$-level of the membership function, i.e.: $\mu(\underline{q}_\alpha)=\mu(\overline{q}_\alpha)=\alpha,\ \ \mu(\underline{q})=\mu(\overline{q})=0$.

**Definition 3**. The l(X)-form of the uncertain parameter q is the triple

$$\tilde{q}=<\underline{q},\overline{q},l(x)>$$

where l(X) is the expert knowledge base in the form of systems of fuzzy logical propositions:

$$\text{if } \left(x_1=a_1^{j1}\right) \text{ and } \left(x_2=a_2^{j1}\right) \text{ and} \ldots \text{ and } \left(x_n=a_n^{j1}\right) \text{ or } \ldots$$

$$\text{if } \left(x_1=a_1^{jk_j}\right) \text{ and } \left(x_2=a_2^{jk_j}\right) \text{ and} \ldots \text{ and } \left(x_n=a_n^{jk_j}\right), \text{ then } l=l_j,$$

where $a_i^{jp}=\int_{U_i}\mu^{jp}(x_i)/x_i, i=\overline{1,n}, p=\overline{1,k_j}$, where $k_j$ is the number of fuzzy rules for $l=l_j$, and

$\mu^{ip}(x_i)$ is the membership function of the variable $x_i\in U_i$ to the fuzzy term $a_i^{jp}$ estimating the factor $x_i$ in rule with number jp, $i=\overline{1,n}, j=\overline{1,m},\ p=\overline{1,k_j}$.

L(X)- form ties the level l of the parameter $q\in[\underline{q},\overline{q}]$ with the vector of influential factors $X=(x_1,x_2,...,x_n)$. The l(x)-form is transformed into l-form by fuzzy inference (Zimmerman, 1996). Transition from l-form to $\alpha$ -form is carried out via the membership function of fuzzy number.

## 5. Extending the reliability models to the fuzzy case

**Definition 4.** *Extension principle* (Zimmerman, 1996). If the function $y = f(q_1, q_2, ..., q_n)$ of n independent variables is given and its arguments $q_i$ are fuzzy numbers $\tilde{q}_i$ in $\alpha$ -form (4) ( $i = \overline{1,n}$ *),* then the value of the function $\tilde{y} = f(\tilde{q}_1, \tilde{q}_2, ..., \tilde{q}_n)$ is the fuzzy number $\tilde{y}$ represented in $\alpha$ -form:

$$\tilde{y} = \bigcup_{\alpha \in [0,1]} \left( \underline{y}_\alpha, \overline{y}_\alpha \right)$$

where $\underline{y}_\alpha = \inf\limits_{q_{i_\alpha} \in [\underline{q}_{i_\alpha}, \overline{q}_{i_\alpha}],\ i=\overline{1,n}} \left( f(q_{1_\alpha}, q_{2_\alpha}, ..., q_{n_\alpha}) \right)$; $\overline{y}_\alpha = \sup\limits_{q_{i_\alpha} \in [\underline{q}_{i_\alpha}, \overline{q}_{i_\alpha}],\ i=\overline{1,n}} \left( f(q_{1_\alpha}, q_{2_\alpha}, ..., q_{n_\alpha}) \right)$.

The extension principle easily produces fuzzy analogues of reliability models of algorithm execution (1) -(3). An example of application the fuzzy reliability models for assessment probabilistic-time characteristics of a ticket-booking information system is described in (Rotshtein and Shtovba, 1998).

## 6. Conclusions

The main obstacle to the application of probabilistic reliability models is the absence of input data that reflect real-life conditions describing the operation of the system. The method proposed in this paper for estimating the reliability of algorithms is one of the formal approaches to resolving the difficulty with source data by means of linguistic expert information and fuzzy extension principle. Contrary to semi-Markov models used in reliability theory, the proposed technique is free from time-consumed procedures for convolution of the distribution functions of the system sojourn time in a given state.

## 7. References


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