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ARTIFICIAL INTELLIGENCE

Fuzzy Multicriteria Analysis of Variants with the Use of Paired Comparisons

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Abstract—In this paper, a variant ranking method based on the linguistic evaluations of partial criteria is proposed. The ranks of partial criteria, as well as the membership functions of qualitative estimations of each variant, are determined on the basis of paired comparisons by the Saaty scale. The choice of the best variant is performed by the Bellman–Zadeh principle. The theoretical principles of the method are illustrated by an example of the comparison of the technical and economic levels of innovation projects.

INTRODUCTION

A multicriteria analysis of variants is an important decision problem that arises not only in engineering, but also in economics, education, politics, etc. The known methods of multicriteria analysis that are used in engineering systems [1] suggest that the vector of partial criteria that estimates the system should be transformed into a scalar integral criterion. A major drawback of this method is that it is poorly adapted to the qualitative criteria that are estimated by expert methods.

The theory of fuzzy sets serves as one of the possible ways to formalize expert evaluations of criteria. In this case, the integral criterion is considered as a fuzzy convolution of partial criteria [2]. A disadvantage of this method is associated with the fact that, sometimes, an expert can barely evaluate a certain partial criterion, even on a qualitative level. Note that, in this case, it is easier for an expert to determine the better of two variants, i.e., to perform paired comparisons.

The method of the multicriteria analysis of the variants proposed in this paper requires neither a quantitative estimation of partial criteria nor the scalarization procedure. This method uses the available linguistic information about the quality of variants given by the following paired comparisons:

by criterion A, variant 1 is *approximately the same* as variant 2,

by criterion B, variant 1 is *much better* than variant 2, etc.

1. STATEMENT OF THE PROBLEM

Suppose that the following sets are known: $S = \{s_1, s_2, \ldots, s_n\}$, the set of variants (analogues) that are subject to multicriteria analysis, and $C = \{c_1, c_2, \ldots, c_m\}$, the set of quantitative and qualitative criteria by which the variants are estimated. The problem consists in ordering the elements of the set S according to the criteria from the set C. To solve this problem, we propose the following principles:

Principle 1. Interpretation of the criteria as fuzzy sets defined on universes of discourse of variants by membership functions.

Principle 2. Determination of the membership functions of the fuzzy sets on the basis of expert information about paired comparisons of variants by the 9-point Saaty scale.

Principle 3. Ranking of variants on the basis of intersections of fuzzy sets, criteria that correspond to the Bellman–Zadeh scheme known in decision theory.

Principle 4. Ranking of criteria by the method of paired comparisons and interpretation of the ranks obtained as the degree of concentration of the corresponding membership functions.

2. CRITERIA AS FUZZY SETS

Suppose that $\mu^l(s_i)$ is a number from the interval [0, 1] that characterizes the estimation level of the variant $s_i \in S$ by the criterion $c_l \in C$: the greater the number $\mu^l(x_i)$, the higher the estimate of the variant by the criterion $c_l \in C$, $i = \overline{1, n}$, $l = \overline{1, m}$. Then, the criterion $c_l \in C$ can be represented as a fuzzy set $\tilde{c}_l$ defined on a umverse of discourse S so that

$$\tilde{c}_l = \left\{ \frac{\mu^l(s_1)}{s_1}, \frac{\mu^l(s_2)}{s_2}, \ldots, \frac{\mu^l(s_n)}{s_n} \right\}, \qquad (2.1)$$

where $\mu^l(s_i)$ is the grade of membership of the element s_i in the fuzzy set $\tilde{c}_l$.

To determine the grades of membership involved in Eq. (2.1), we compose the matrices of paired comparisons of variants with respect to every criterion. The total number of such matrices coincides with the number of criteria and is equal to m

For the criteria $c_l \in C$, the matrix of paired comparisons has the form

$$A(c_l) = \begin{array}{c} \\ s_1 \\ s_2 \\ \vdots \\ s_n \end{array} \begin{array}{c} \begin{array}{cccc} s_1 & s_2 & \dots & s_n \end{array} \\ \left[\begin{array}{cccc} a_{11}^l & a_{12}^l & \dots & a_{2n}^l \\ a_{21}^l & a_{22}^l & \dots & a_{2n}^l \\ \dots & \dots & \dots & \dots \\ a_{n1}^l & a_{n2}^l & \dots & a_{nn}^l \end{array}\right] \end{array}, \qquad (2.2)$$

where the elements a_{ij}^l are estimated by an expert by the 9-point Saaty scale [3]: 1, if variant s_i has *no advantage* over variant s_j; 3, if variant s_i has a *small advantage* over s_j; 5, if variant s_i has a *significant advantage* over s_j; 7, if variant s_i has a *clear advantage* over s_j; 9, if variant s_i has an *absolute advantage* over s_j; and 2, 4, 6, and 8 are *intermediate* comparative estimates.

If matrix (2.2) is known, it allows one to rank each variant $s_i \in S$ with respect to each criterion $c_l \in C$ by using the Saaty method. To calculate the ranks by the method that was first proposed in [3] and later developed in [4], one should find the eigenvector of matrix (2.2). At a first approximation, one can suggest that the paired comparisons are consistent, i.e., that matrix (2.2) has the following properties:

it is diagonal, i.e., $a_{ii}^l = 1$, $i = \overline{1, n}$;

it is inversely symmetric; i.e., the elements symmetric with respect to the main diagonal are related by

$$a_{ij}^l = 1/a_{ji}^l;$$

it is transitive; i.e., $a_{ik}^l a_{kj}^l = a_{ij}^l$.

These properties allow one to determine all the elements of matrix (2.2) provided that $n - 1$ off-diagonal elements are known. For example, if the kth row, i.e., the elements a_{kj}^l ($j = \overline{1, n}$) are known, then an arbitrary element a_{ij}^l is determined as

$$a_{ij}^l = \frac{a_{kj}^l}{a_{ki}^l}, \quad i, j, k = \overline{1, n}, \quad l = \overline{1, m}.$$

When all the elements of matrix (2.2) are determined, the grades of membership that are necessary for forming the fuzzy set (2.1) are calculated by the formula [5]

$$\mu^l(s_i) = \frac{1}{a_{1i}^l + a_{2i}^l + \dots + a_{ni}^l}. \qquad (2.3)$$

In contrast to the Saaty method, formula (2.3) does not involve the laborious computational procedures associated with determining the eigenvector of matrix (2.2). However, this formula only can be used for consistent paired comparisons. When the matrix of paired comparisons is not inversely symmetric and transitive, the ranks of the elements are determined by the method of analysis of hierarchies [4].

3. EQUILIBRIUM CRITERIA

Based on the Bellman–Zadeh principle [6], we consider, as the best variant, the one that is the best simultaneously with respect to the criteria $c_1, c_2, \dots, c_m$. Therefore, the fuzzy set $\tilde{D}$ necessary for the rating analysis is determined as the intersection (integral estimation criterion of a variant)

$$\tilde{D} = \tilde{c}_1 \cap \tilde{c}_2 \cap \dots \cap \tilde{c}_m.$$

Taking into account that the operation of intersection in the theory of fuzzy sets corresponds to the operation min, we have

$$\tilde{D} = \left\{ \frac{\min\limits_{l=\overline{1,m}}\left[\mu^l(s_1)\right]}{s_1}, \frac{\min\limits_{l=\overline{1,m}}\left[\mu^l(s_2)\right]}{s_2}, \dots, \frac{\min\limits_{l=\overline{1,m}}\left[\mu^l(s_n)\right]}{s_n} \right\}. \qquad (3.1)$$

The analysis of the fuzzy set obtained shows that the variant for which the grade of membership (numerator) is maximal should be considered as the best one.

4. NONEQUILIBRIUM CRITERIA

Let $w_1, w_2, \dots, w_m$ be the coefficients of relative importance (or the ranks) of the criteria $c_1, c_2, \dots, c_m$ such that $w_1 + w_2 + \dots + w_m = 1$. To determine the coefficients w_l, $l = \overline{1, m}$, we have to construct a matrix of paired comparisons of the criteria $c_l \in C$ that is similar to (2.2) and apply formula (2.3).

The method of decision making with regard to the importance coefficients should guarantee an increase in the difference between the variants with respect to the most important criteria and a decrease in the difference with respect to the least important criteria. To take into account the importance coefficients, we use the idea that underlies the operations of concentration and dilatation of fuzzy sets [7]. These operations consist in transforming a fuzzy set by raising the membership function to a positive power: to the power 2 for the concentration and 1/2 for the dilatation. In the general case, the greater the exponent, the greater the increase in the difference between the elements of the fuzzy set, i.e., the fuzzy set becomes more concentrated. Thus, formula (3.1) is rewritten as

$$\tilde{D} = \left\{ \frac{\min_{l=1,m}\left[\mu^l(s_1)\right]^{w_l}}{s_1}, \frac{\min_{l=1,m}\left[\mu^l(s_2)\right]^{w_l}}{s_2}, \ldots, \frac{\min_{l=1,m}\left[\mu^l(s_n)\right]^{w_l}}{s_n} \right\} \qquad (4.1)$$

where the exponent w_l characterizes the concentration of the fuzzy set $\tilde{c}_l$ according to the importance level of the criterion $c_l \in C$.

5. AN EXAMPLE OF MULTICRITERIA ANALYSIS

As an example illustrating the application of the method proposed, consider the comparison of the technical and economic levels of three projects (s_1, s_2, s_3) applied to the innovation foundation for a grant.

5.1. Estimation Criteria of Variants

To assess the technical and economic levels of the projects, we use the following criteria: c_1, the *scale of a project*; c_2, the *novelty of a project*; c_3, the *priority of the direction*; c_4, the *level of development*; c_5, *legal immunity*; and c_6, *ecological level*.

5.2. Paired Comparisons

The comparison of the projects s_1, s_2, and s_3 with respect to the criteria c_1–c_6 yields the linguistic propositions given in the table.

5.3. Matrices of Paired Comparisons

The expert judgments presented in the table correspond to the following matrices of paired comparisons:

$$A(c_1) = \begin{matrix} & s_1 & s_2 & s_3 \\ s_1 & 1 & \mathbf{1} & 0.2 \\ s_2 & 1 & 1 & 0.2 \\ s_3 & \mathbf{5} & 5 & 1 \end{matrix};$$

$$A(c_2) = \begin{matrix} & s_1 & s_2 & s_3 \\ s_1 & 1 & 1.33 & \mathbf{4} \\ s_2 & 0.75 & 1 & \mathbf{3} \\ s_3 & 0.25 & 0.33 & 1 \end{matrix};$$

$$A(c_3) = \begin{matrix} & s_1 & s_2 & s_3 \\ s_1 & 1 & \mathbf{5} & \mathbf{7} \\ s_2 & 0.2 & 1 & 1.4 \\ s_3 & 0.14 & 0.71 & 1 \end{matrix};$$

$$A(c_4) = \begin{matrix} & s_1 & s_2 & s_3 \\ s_1 & 1 & 0.33 & 0.5 \\ s_2 & \mathbf{3} & 1 & 1.5 \\ s_3 & \mathbf{2} & 0.67 & 1 \end{matrix};$$

$$A(c_5) = \begin{matrix} & s_1 & s_2 & s_3 \\ s_1 & 1 & \mathbf{5} & \mathbf{6} \\ s_2 & 0.2 & 1 & 1.2 \\ s_3 & 0.17 & 0.83 & 1 \end{matrix};$$

$$A(c_6) = \begin{matrix} & s_1 & s_2 & s_3 \\ s_1 & 1 & \mathbf{4} & 0.5 \\ s_2 & 0.25 & 1 & 0.13 \\ s_3 & \mathbf{2} & 8 & 1 \end{matrix}.$$

In these matrices, the elements corresponding to the paired comparisons presented in the table are heavily drowned. The other elements are obtained by using the fact that the matrix of paired comparisons is diagonal and possesses the properties of transitivity and inverse symmetry pointed out in Section 2.

Paired comparison of projects

Criteria	Paired comparison
c_1	No advantage of s_1 over s_2
	Significant advantage of s_3 over s_1
c_2	An almost significant advantage of s_1 over s_3
	A small advantage of s_2 over s_3
c_3	Significant advantage of s_1 over s_2
	A clear advantage of s_1 over s_3
c_4	A small advantage of s_2 over s_1
	An almost small advantage of s_3 over s_1
c_5	Significant advantage of s_1 over s_2
	An almost clear advantage of s_1 over s_3
c_6	An almost significant advantage of s_1 over s_2
	An almost significant advantage of s_3 over s_1

5.4. Criteria as Fuzzy Sets

Using the matrices of paired comparisons and formula (2.3), we obtain

$$\tilde{c}_1 = \left\{\frac{0.14}{s_1}, \frac{0.14}{s_2}, \frac{0.72}{s_3}\right\}, \quad \tilde{c}_2 = \left\{\frac{0.5}{s_1}, \frac{0.38}{s_2}, \frac{0.12}{s_3}\right\},$$

$$\tilde{c}_3 = \left\{\frac{0.74}{s_1}, \frac{0.15}{s_2}, \frac{0.11}{s_3}\right\}, \quad \tilde{c}_4 = \left\{\frac{0.17}{s_1}, \frac{0.5}{s_2}, \frac{0.33}{s_3}\right\},$$

$$\tilde{c}_5 = \left\{\frac{0.73}{s_1}, \frac{0.15}{s_2}, \frac{0.12}{s_3}\right\}, \quad \tilde{c}_6 = \left\{\frac{0.31}{s_1}, \frac{0.08}{s_2}, \frac{0.61}{s_3}\right\}.$$

5.5. The Case of Equilibrium Criteria

Using the fuzzy sets c_1–c_6 and model (3.1), we obtain

$$\tilde{D} = \left\{\frac{0.14}{s_1}, \frac{0.08}{s_2}, \frac{0.11}{s_3}\right\},$$

which demonstrates that the project s_1 has a significant advantage over the project s_2 and a small advantage over the project s_3.

5.6. The Case of Nonequilibrium Criteria

Determine the ranks of the criteria c_1–c_6 using the following linguistic propositions:

an *almost significant* advantage of c_2 over c_1;

a *clear* advantage of c_3 over c_1;

a *small* advantage of c_3 over c_5;

an *almost small* advantage of c_4 over c_6;

no advantage of c_5 over c_6.

These expert judgments correspond to the following matrix of paired comparisons:

$$W = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{array} \begin{array}{c} \begin{array}{cccccc} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \end{array} \\ \begin{bmatrix} 1 & 0.25 & 0.14 & 0.21 & 0.43 & 0.43 \\ \mathbf{4} & 1 & 0.57 & 0.86 & 1.71 & 1.71 \\ \mathbf{7} & 1.75 & 1 & 1.5 & \mathbf{3} & 3 \\ 4.67 & 1.17 & 0.67 & 1 & 2 & \mathbf{2} \\ 2.33 & 0.58 & 0.33 & 0.5 & 1 & \mathbf{1} \\ 2.33 & 0.58 & 0.33 & 0.5 & 1 & 1 \end{bmatrix} \end{array}.$$

Applying formula (2.3), determine the ranks of the criteria c_1–c_6:

$$w_1 = 0.04; \quad w_2 = 0.19;$$

$$w_3 = 0.33; \quad w_4 = 0.22;$$

$$w_5 = 0.11; \quad w_6 = 0.11,$$

this means that the priority of the direction (c_3) and the level of the development of the project (c_4) are the most important criteria. Then, according to (4.1), we obtain

$$\tilde{c}_1 = \left\{\frac{0.14^{0.04}}{s_1}, \frac{0.14^{0.04}}{s_2}, \frac{0.72^{0.04}}{s_3}\right\} = \left\{\frac{0.91}{s_1}, \frac{0.91}{s_2}, \frac{0.98}{s_3}\right\};$$

$$\tilde{c}_2 = \left\{\frac{0.5^{0.19}}{s_1}, \frac{0.38^{0.19}}{s_2}, \frac{0.12^{0.19}}{s_3}\right\} = \left\{\frac{0.88}{s_1}, \frac{0.83}{s_2}, \frac{0.68}{s_3}\right\};$$

$$\tilde{c}_3 = \left\{\frac{0.74^{0.33}}{s_1}, \frac{0.15^{0.33}}{s_2}, \frac{0.11^{0.33}}{s_3}\right\} = \left\{\frac{0.91}{s_1}, \frac{0.53}{s_2}, \frac{0.48}{s_3}\right\};$$

$$\tilde{c}_4 = \left\{\frac{0.17^{0.22}}{s_1}, \frac{0.5^{0.22}}{s_2}, \frac{0.33^{0.22}}{s_3}\right\} = \left\{\frac{0.68}{s_1}, \frac{0.86}{s_2}, \frac{0.79}{s_3}\right\};$$

$$\tilde{c}_5 = \left\{\frac{0.73^{0.11}}{s_1}, \frac{0.15^{0.11}}{s_2}, \frac{0.12^{0.11}}{s_3}\right\} = \left\{\frac{0.97}{s_1}, \frac{0.81}{s_2}, \frac{0.79}{s_3}\right\};$$

$$\tilde{c}_6 = \left\{\frac{0.31^{0.11}}{s_1}, \frac{0.08^{0.11}}{s_2}, \frac{0.61^{0.11}}{s_3}\right\} = \left\{\frac{0.88}{s_1}, \frac{0.76}{s_2}, \frac{0.95}{s_3}\right\}.$$

The operation of the intersection of the fuzzy sets c_1–c_6 yields

$$\tilde{D} = \left\{\frac{0.68}{s_1}, \frac{0.53}{s_2}, \frac{0.48}{s_3}\right\},$$

which shows that the project s_1 has a significant advantage over the projects s_2 and s_3 and that project s_2 has a small advantage over s_3.

CONCLUSION

The method proposed in this paper allows one to carry out a multicriteria analysis of variants on the basis of paired comparisons. The use of paired comparisons instead of absolute values of the criteria is more convenient for experts. A characteristic feature of the method proposed is the application of the Bellman–Zadeh principle, which allows one to choose a variant that maximally satisfies all the criteria simultaneously. The method proposed can be used for the multicriteria analysis of variants in the decision problems in engineering, economics, politics, education, medicine, and other fields.

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